

The Bondage Number of Mesh Networks *

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Abstract: The bondage number $b(G)$ of a nonempty graph G is the smallest number of edges whose removal from G results in a graph with domination number greater than that of G . Denote $P_n \times P_m$ be the Cartesian product of two paths P_n and P_m . This paper determines that the exact value of $b(P_n \times P_2)$, $b(P_n \times P_3)$ and $b(P_n \times P_4)$ for $n \geq 2$.

Keywords: bondage number, dominating set, domination number, mesh networks

AMS Subject Classification: 05C25, 05C40, 05C12, 05C69

1 Introduction

Throughout this paper, for the terminology and notation not defined here, we refer the reader to [29, 30]. A graph $G = (V, E)$ is considered as an undirected and simple graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set.

A nonempty subset $D \subseteq V(G)$ is said a dominating set in G if every vertex in G is either in D or adjacent to a vertex in D . The domination number $\gamma(G)$ of G is the minimum cardinality of all dominating sets in G . A dominating set D is said to be the minimum if $|D| = \gamma(G)$. The bondage number $b(G)$ of a nonempty graph G is the minimum number of edges whose removal from G results in a graph with larger domination number, that is,

$$b(G) = \min\{|B| : B \subseteq E(G), \gamma(G - B) > \gamma(G)\}.$$

A nonempty subset $B \subseteq E(G)$ is said a bondage set of G if $\gamma(G - B) > \gamma(G)$.

The concept of the bondage number is proposed by Fink et al. [8] for an undirected graph and by Carlson and Develin [5] for a digraph. However the first result on bondage numbers is obtained by Bauer et al. [1]. There are many research articles on the bondage number for undirected graphs and digraphs (see, for example [1]~[2], [5]~[17], [19, 20], [22]~[28], [31]). In particular, Hu and Xu [13] have showed that the problem determining bondage number for general graphs is NP-hard.

*The work was supported by NNSF of China (No. 11071233).

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Apart from its own theoretical interest, the study on the bondage number is also motivated by the increasing importance in the design and analysis of interconnection networks. It is well-known that the topological structure of an interconnection network can be modeled by a connected graph whose vertices represent sites of the network and whose edges represent physical communication links. A minimum dominating set in the graph corresponds to a smallest set of sites selected in the network for some particular uses, such as placing transmitters. Such a set may not work when some communication links happen fault. The fault is possible in real world (hacking, experimental error, terrorism, etc), so one needs to consider it. What is the minimum number of faulty links which will make all minimum dominating sets of the original network not work any more? Such a minimum number is the bondage number, which measures the robustness of a network with respect to link failures, wherever a minimum dominating set is required for some applications.

Motivated by the above relevance of bondage number, one wants to know how to compute it for a network. However, this computation is generally difficult; no efficient algorithm has been proposed as yet. Therefore, it is of significance to develop a technique to determine bondage numbers for some special graphs or networks. However, the exact value of the bondage number has been determined for only a few classes of graphs, such as complete graphs, paths, cycles and complete t -partition graphs (see, Fink *et al.* [8] for the undirected cases, Huang and Xu [14], Zhang *et al.* [31] for the directed cases), trees (see, Bauer *et al.* [1], Hartnell and Rall [10], Hartnell *et al.* [9], Topp and Vestergaard [28], Teschner [26]), de Bruijn and Kautz digraphs (see, Huang and Xu [14]).

Let P_n and C_n be a path and a cycle of order n , respectively. For the Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 , Dunbar *et al.* [7] determined $b(C_n \times P_2)$ for $n \geq 3$, Sohn, Yuan and Jeong [23] determined that $b(C_n \times C_3)$ for $n \geq 4$, Kang, Sohn and Kim [19] determined $b(C_n \times C_4)$ for $n \geq 4$, Huang and Xu [17] presented that $b(C_{5i} \times C_{5j})$ for any positive integers i and j ; Cao, Yuan and Moo [6] determined $b(C_n \times C_5)$ for $n \geq 5$ and $n \not\equiv 3 \pmod{5}$, but $b(C_n \times C_m)$ for $m \geq 6$ has been not determined as yet.

The mesh $P_n \times P_m$ is a very famous network, and its domination number has been determined when $1 \leq m \leq 6$ for many years [3, 4, 18, 21]. However, its bondage number has been not determined as yet. For $n = 1$, $P_1 \times P_m$ is isomorphic to P_m , and $b(P_m)$ has been determined. In this paper, we present the exact value of $b(P_n \times P_2)$, $b(P_n \times P_3)$ and $b(P_n \times P_4)$ for $n \geq 2$.

The rest of the paper is organized as follows. Section 2 presents some useful results, Section 3 determines $b(P_n \times P_2)$, Section 4 determines $b(P_n \times P_3)$, and Section 5 determines $b(P_n \times P_4)$. Some remarks are in Section 6, in which we propose a conjecture: $b(P_n \times P_m) \leq 2$ for $m \geq 5$.

2 Preliminary results

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. The *Cartesian product* of G_1 and G_2 is an undirected graph, denoted by $G_1 \times G_2$, where $V(G_1 \times G_2) = V_1 \times V_2$, two distinct vertices x_1x_2 and y_1y_2 , where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, are linked by an edge in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, such an edge is called a *vertical edge*, or $x_2 = y_2$ and $x_1y_1 \in E(G_1)$, such an edge is called a *horizontal edge*. It is clear, as a graphic operation, that the Cartesian product satisfies commutative associative law if identify isomorphic graphs.

Throughout this paper, the notation P_n denotes a path with vertex-set $\{1, 2, \dots, n\}$. The (n, m) -mesh network, denoted by $G_{n,m}$, is defined as the Cartesian product $P_n \times P_m$, with the vertex-set $\{u_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

The graph shown in Figure 1 is a $(4, 3)$ -mesh network $G_{4,3}$. It is clear, as a graphic operation, that the cartesian product satisfies commutative associative law if we identify isomorphic graphs, that is, $G_{n,m} \cong G_{m,n}$.

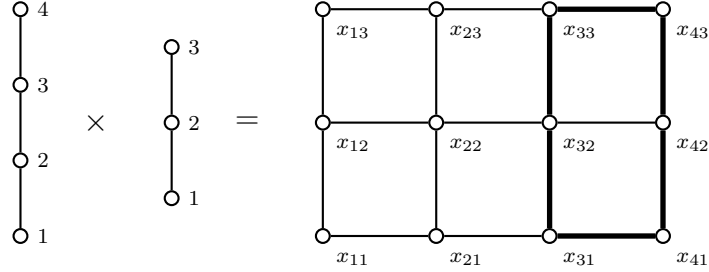


Figure 1: A $(4, 3)$ -mesh network $G_{4,3} = P_4 \times P_3$

The following notations will be used in this paper. For a positive integer t with $t < n$, $G_{t,m}$ is a subgraph of $G_{n,m}$. Denote $H_{n-t,m} = G_{n,m} - G_{t,m}$, that is, $H_{n-t,m}$ is a subgraph of $G_{n,m}$ induced by the set of vertices $\{u_{i,j} \mid t+1 \leq i \leq n, 1 \leq j \leq m\}$. Clearly, $H_{n-t,m} \cong G_{n-t,m}$. The graph shown in Figure 1 by heavy lines is a subgraph $H_{2,3}$ of $G_{4,3}$, where $n = 4, t = 2$ and $m = 3$ is a such example.

Note that both $G_{0,m}$ and $H_{n-n,m}$ are nominal graphs. For convenience of statements, we allow $G_{0,m}$ and $H_{n-n,m}$ to appear in this paper. If so, we specify consider that their total dominating sets are empty.

In Addition, let $Y_i = \{u_{i,j} : j = 1, 2, \dots, m\}$ for each $i = 1, 2, \dots, n$, called a set of vertical vertices of i in $G_{n,m}$.

We state some useful results on $\gamma(G_{n,m})$ to be used in this paper.

Lemma 2.1 [18, 21] *Let P_n and C_m be a path and a cycle of order $n \geq 1$ and $m \geq 3$, respectively. Then*

$$\begin{aligned} \gamma(G_{n,2}) &= \lceil \frac{n+1}{2} \rceil; \\ \gamma(G_{n,3}) &= n - \lfloor \frac{n-1}{4} \rfloor; \\ \gamma(G_{n,4}) &= \begin{cases} n+1, & \text{if } n = 1, 2, 3, 5, 6 \text{ or } 9 \\ n, & \text{otherwise;} \end{cases} \\ \gamma(C_m \times C_3) &= n - \lfloor \frac{n}{4} \rfloor. \end{aligned}$$

Lemma 2.2 *Let D be a dominating set of $G_{n,m}$. Then $\gamma(G_{i,m}) \leq |D \cap V(G_{i+1,m})|$ and $\gamma(H_{n-i,m}) \leq |D \cap V(H_{n-i+1,m})|$ for each $i = 1, 2, \dots, n-1$ and $m \geq 2$.*

Proof. We only need to prove that $\gamma(G_{i,m}) \leq |D \cap V(G_{i+1,m})|$ since $H_{n-i,m} \cong G_{n-i,m}$. Let $D' = D \cap V(G_{i+1,m})$.

If $D' \cap Y_{i+1} = \emptyset$, then D' is a dominating set of $G_{i,m}$, and hence $\gamma(G_{i,m}) \leq |D'|$.

Assume $D' \cap Y_{i+1} \neq \emptyset$ below. Let $B_i = \{j \mid u_{i+1,j} \in D'\}$. Then $D'' = (D' \setminus Y_{i+1}) \cup \{u_{i,j} \mid j \in B_i\}$ is a dominating set of $G_{i,m}$ and $|D''| \leq |D'|$. Thus, we have $\gamma(G_{i,m}) \leq |D''| \leq |D'|$. The lemma follows. \blacksquare

3 The bondage number of $G_{n,2}$

Theorem 3.1 $b(G_{2,2}) = 3$, $b(G_{3,2}) = 2$, and $b(G_{n,2}) = 1$ if n is odd and $b(G_{n,2}) = 2$ if n is even for $n \geq 4$.

Proof. It is easy to verify that $b(G_{2,2}) = 3$ and $b(G_{3,2}) = 2$. In the following, consider $n \geq 4$. When n is odd, we consider the domination number of $G = G_{n,2} - u_{1,1}u_{1,2}$. Let D be a minimum dominating set of G . Then either $u_{1,1} \in D$ or $u_{1,2} \in D$, and either $u_{1,2} \in D$ or $u_{2,2} \in D$. Without loss of generality we assume that $u_{1,1}, u_{1,2} \notin D$ and $u_{2,1}, u_{2,2} \in D$. By Lemma 2.2, $|D \cap V(H_{n-2,m})| \geq \gamma(H_{n-3,2})$. Then by Lemma 2.1, $|D| \geq 2 + \gamma(H_{n-3,2}) = 2 + \lceil \frac{n-3+1}{2} \rceil = 1 + \gamma(G_{n,2})$, which yields $b(G_{n,2}) = 1$.

When n is even, we claim that $\gamma(G_{n,2}) = \gamma(G_{n,2} - e)$ for any $e \in E(G_{n,2})$.

To prove this claim, we first consider that e is a vertical edge, and let $e = u_{j,1}u_{j,2}$.

If j is even, then all the vertices $u_{i,1}, i \equiv 1 \pmod{4}$, $u_{i,2}, i \equiv 3 \pmod{4}$, $u_{n,1}$ if $n \equiv 0 \pmod{4}$ or $u_{n,2}$ if $n \equiv 2 \pmod{4}$, form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

If j is odd, then all the vertices $u_{i,1}, i \equiv 2 \pmod{4}$, $u_{i,2}, i \equiv 0 \pmod{4}$ and $u_{2,2}$ form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

Assume now that e is a horizontal edge. Without loss of generality, let $e = u_{j,1}u_{j+1,1}$.

If $j \equiv 2$ or $3 \pmod{4}$, then all the vertices $u_{i,1}, i \equiv 1 \pmod{4}$, $u_{i,2}, i \equiv 3 \pmod{4}$, and $u_{n,1}$ form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

If $j \equiv 0$ or $1 \pmod{4}$, then all the vertices $u_{i,2}, i \equiv 1 \pmod{4}$, $u_{i,1}, i \equiv 3 \pmod{4}$, and $u_{n,1}$ form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

So we have that $b(G_{n,2}) \geq 2$. Next, we show that $b(G_{n,2}) \leq 2$. Let $e_1 = u_{2,1}u_{3,1}$, $e_2 = u_{2,2}u_{3,2}$, and $G' = G_{n,2} - \{e_1, e_2\}$. Then G' consists of two connected components, one is $G_{2,2}$ and the other one is $H_{n-2,2}$. By Lemma 2.1, we have $\gamma(G') = \gamma(G_{2,2}) + \gamma(H_{n-2,2}) = 2 + \lceil \frac{n-2+1}{2} \rceil = 1 + \gamma(G_{n,2})$, which implies $b(G_{n,2}) \leq 2$. Thus $b(G_{n,2}) = 2$. ■

4 The bondage number of $G_{n,3}$

Proposition 4.1 ([18]) *A minimum dominating set D of $G_{n,3}$ is constructed as follows.*

$$D = \begin{cases} \{u_{i,2} : i \equiv 1 \pmod{4}\} \cup \{u_{i,1}, u_{i,3} : i \equiv 3 \pmod{4}\} & \text{if } n \text{ is odd,} \\ \{u_{i,2} : i \equiv 1 \pmod{4}\} \cup \{u_{i,1}, u_{i,3} : i \equiv 3 \pmod{4}\} \cup \{u_{n,2}\} & \text{if } n \text{ is even.} \end{cases}$$

Lemma 4.1 $\gamma(G_{n,3} - u_{1,j}) \geq \gamma(G_{n,3}) = n - \lfloor \frac{n-1}{4} \rfloor$ for each $j = 1, 2, 3$ and $n \equiv 1, 2$ or $3 \pmod{4}$.

Proof. It is easy to verify that the conclusion is true for $n = 1, 2, 3$. In the following, assume $n \geq 4$. Let $G = G_{n,3} - u_{1,j}$ and D be a minimum dominating set of G . we only need to show $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$.

If $(Y_1 - u_{1,j}) \cap D \neq \emptyset$, then D is a dominating set of $C_n \times C_3$. By Lemma 2.1, $|D| \geq \gamma(C_n \times C_3) = n - \lfloor \frac{n}{4} \rfloor = n - \lfloor \frac{n-1}{4} \rfloor$.

If $(Y_1 - u_{1,j}) \cap D = \emptyset$, then $|Y_2 \cap D| \geq 2$. By Lemma 2.2, $|D \cap V(H_{n-2,3})| \geq \gamma(H_{n-3,3})$. By Lemma 2.1, $|D| \geq 2 + \gamma(H_{n-3,3}) = 2 + n - 3 - \lfloor \frac{n-3-1}{4} \rfloor = n - \lfloor \frac{n-1}{4} \rfloor$, as required. ■

Lemma 4.2 $\gamma(G_{n,3} - u_{1,1}) \geq \gamma(G_{n,3}) = n - \lfloor \frac{n-1}{4} \rfloor$ for $n \equiv 0 \pmod{4}$.

Proof. Let D be a minimum dominating set of $G_{n,3} - u_{1,1}$. we only need to prove $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$. It is easy to verify that the assertion is true for $n = 4$. In the following, we consider the case $n \geq 8$. We consider the following three cases, respectively.

Case 1 $u_{1,2} \in D$ or $u_{2,1} \in D$.

In this case, D is also a dominating set of $G_{n,3}$, and so $|D| \geq \gamma(G_{n,3}) = n - \lfloor \frac{n-1}{4} \rfloor$.

Case 2 $u_{1,2} \notin D$, $u_{2,1} \notin D$ and $u_{1,3} \in D$.

In this case, $D \setminus \{u_{1,3}\}$ is a dominating set of $H_{n-1,3}$ or $H_{n-1,3} - u_{2,3}$. By Lemma 4.1, $|D \setminus \{u_{1,3}\}| \geq n - 1 - \lfloor \frac{n-1-1}{4} \rfloor$, and so $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$.

Case 3 $u_{1,2} \notin D$, $u_{2,1} \notin D$ and $u_{1,3} \notin D$.

In this case, $u_{2,2}, u_{2,3} \in D$. We prove the conclusion by two subcases.

Subcase 3.1 $Y_3 \cap D \neq \emptyset$.

Then $D \setminus \{u_{2,2}, u_{2,3}\}$ is a dominating set of $H_{n-2,3}$ or $H_{n-2,3} - u_{3,1}$ or $H_{n-2,3} - u_{3,3}$. By Lemma 4.1, $|D \setminus \{u_{2,2}, u_{2,3}\}| \geq n - 2 - \lfloor \frac{n-2-1}{4} \rfloor$. Thus, $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$.

Subcase 3.2 $Y_3 \cap D = \emptyset$.

Then $u_{4,1} \in D$.

If $u_{4,2} \in D$ or $u_{4,3} \in D$, then $D \setminus \{u_{2,2}, u_{2,3}\}$ is a dominating set of $H_{n-2,3}$ or $H_{n-2,3} - u_{3,2}$ or $H_{n-2,3} - u_{3,3}$. By Lemma 4.1, $|D \setminus \{u_{2,2}, u_{2,3}\}| \geq n - 2 - \lfloor \frac{n-2-1}{4} \rfloor$. Thus, $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$.

Next, assume $u_{4,2}, u_{4,3} \notin D$. Then $u_{5,3} \in D$. If $u_{5,1} \in D$ or $u_{5,2} \in D$, then $D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}\}$ is a dominating set of $H_{n-4,3}$ and hence $|D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}\}| \geq n - 4 - \lfloor \frac{n-4-1}{4} \rfloor$. Thus, $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$.

If $u_{5,1}, u_{5,2} \notin D$, then $D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}, u_{5,3}\}$ is a dominating set of $H_{n-5,3}$ or $H_{n-5,3} - u_{6,3}$. By Lemma 4.1, $|D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}, u_{5,3}\}| \geq n - 5 - \lfloor \frac{n-5-1}{4} \rfloor$. Thus, $|D| \geq n - \lfloor \frac{n-1}{4} \rfloor$.

The lemma follows. ■

Corollary 4.1 $b(G_{n,3}) \leq 2$.

Proof. By Lemma 4.1 and Lemma 4.2, we have that $\gamma(G_{n,3} - \{u_{1,1}u_{2,1}, u_{1,1}u_{1,2}\}) > \gamma(G_{n,3})$. ■

Lemma 4.3 $b(G_{n,3}) = 1$ for $n \equiv 1$ or $2 \pmod{4}$ and $n \geq 4$.

Proof. Let D be a minimum dominating set of $G_{n,3} - u_{3,1}u_{4,1}$. We only need to prove that $|D| \geq 1 + \gamma(G_{n,3})$ by considering three cases, respectively.

Case 1 $u_{3,2} \in D$ and $u_{3,3} \in D$.

In this case, $|V(G_{3,3}) \cap D| = 4$. By Lemma 2.2, $|D \cap V(H_{n-3,3})| \geq \gamma(H_{n-4,3})$. By Lemma 2.1, $|D| \geq 4 + \gamma(H_{n-4,3}) = 4 + n - 4 - \lfloor \frac{n-4-1}{4} \rfloor = 1 + \gamma(G_{n,3})$.

Case 2 Either $u_{3,2} \in D$ or $u_{3,3} \in D$.

In this case, $|V(G_{3,3}) \cap D| = 3$. Then $D' = D \setminus V(G_{3,3})$ is a dominating set of $H_{n-3,3}$ or $H_{n-3,3} - u_{4,2}$ or $H_{n-3,3} - u_{4,3}$. By Lemma 4.1, $|D| = 3 + |D'| \geq 3 + n - 3 - \lfloor \frac{n-3-1}{4} \rfloor = n + 1 - \lfloor \frac{n-1}{4} \rfloor = 1 + \gamma(G_{n,3})$.

Case 3 $u_{3,2} \notin D$ and $u_{3,3} \notin D$.

In this case, $|V(G_{3,3}) \cap D| = 2$ or $|V(G_{3,3}) \cap D| = 3$. If $|V(G_{3,3}) \cap D| = 3$, then $D \setminus V(G_{3,3})$ is a dominating set of $H_{n-3,3}$. By Lemma 2.1, $|D| \geq 3 + \gamma(H_{n-3,3}) = 3 + n - 3 - \lfloor \frac{n-3-1}{4} \rfloor = 1 + \gamma(G_{n,3})$.

If $|V(G_{3,3}) \cap D| = 2$, then $V(G_{3,3}) \cap D = \{u_{1,3}, u_{2,1}\}$ and $D \setminus V(G_{3,3})$ is a dominating set of $H_{n-2,3} - u_{3,1}$. By Lemma 4.1 or Lemma 4.2, $|D| \geq 2 + n - 2 - \lfloor \frac{n-2-1}{4} \rfloor = n + 1 - \lfloor \frac{n-1}{4} \rfloor = 1 + \gamma(G_{n,3})$.

The lemma follows. ■

Lemma 4.4 $b(G_{n,3}) \geq 2$ for $n \equiv 0 \pmod{4}$.

Proof. By Proposition 4.1, $D = \{u_{i,2} : i \equiv 1 \pmod{4}\} \cup \{u_{i,1}, u_{i,3} : i \equiv 3 \pmod{4}\} \cup \{u_{n,2}\}$ is a minimum dominating set and by the symmetry of $G_{n,3}$, $D' = \{u_{i,2} : i \equiv 0 \pmod{4}\} \cup \{u_{i,1}, u_{i,3} : i \equiv 2 \pmod{4}\} \cup \{u_{1,2}\}$ is also a minimum dominating set. It is clear that if we delete any vertical edge in $G_{n,3}$ or any horizontal edge $u_{i,1}u_{i+1,1}$ and $u_{i,3}u_{i+1,3}$ where $i \equiv 0, 1$ or $3 \pmod{4}$ or any horizontal edge $u_{i,2}u_{i+1,2}$ where $i \equiv 1, 2$ or $3 \pmod{4}$, D or D' is also a domination set. Next, we consider the domination number of $G_{n,3} - e$ where e is an any other edge.

Let $e = u_{i,1}u_{i+1,1}$ or $e = u_{i,3}u_{i+1,3}$ where $i \equiv 2 \pmod{4}$, or $e = u_{i,2}u_{i+1,2}$ where $i \equiv 0 \pmod{4}$. Then $D'' = \{u_{i,1}, u_{i,3} : i \equiv 1 \pmod{4}\} \cup \{u_{i,2} : i \equiv 3 \pmod{4}\} \cup \{u_{n,2}\}$ is a dominating set of $G - e$ with cardinality $n - \lfloor \frac{n-1}{4} \rfloor$. By Lemma 2.1, $|D''| = \gamma(G_{n,3})$.

From the above discussions, $\gamma(G_{n,3} - e) = \gamma(G_{n,3})$ for any edge $e \in E(G_{n,3})$. Thus $b(G_{n,3}) \geq 2$. ■

Lemma 4.5 $b(G_{n,3}) \geq 2$ for $n \equiv 3 \pmod{4}$.

Proof. By Proposition 4.1, $D = \{u_{i,2} : i \equiv 1 \pmod{4}\} \cup \{u_{i,1}, u_{i,3} : i \equiv 3 \pmod{4}\}$ is a minimum dominating set and by the symmetry of $G_{n,3}$, $D' = \{u_{i,2} : i \equiv 3 \pmod{4}\} \cup \{u_{i,1}, u_{i,3} : i \equiv 1 \pmod{4}\}$ is also a minimum dominating set. It is clear that if we delete any edge from $G_{n,3}$, D or D' is also a dominating set. Thus $b(G_{n,3}) \geq 2$. ■

Summing the above results, we have the following theorem, immediately.

Theorem 4.1 For $n \geq 3$, $b(G_{n,3}) = \begin{cases} 1, & \text{if } n \equiv 1 \text{ or } 2 \pmod{4} \\ 2, & \text{if } n \equiv 0 \text{ or } 3 \pmod{4}. \end{cases}$

5 The bondage number of $G_{n,4}$

In this section, let $A = \{1, 2, 3, 5, 6, 9\}$.

Lemma 5.1 Let D be a minimum dominating set of $G_{n,4}$. Then $1 \leq |Y_1 \cap D| \leq 2$ and $1 \leq |Y_n \cap D| \leq 2$ for $n \notin A$.

Proof. By Lemma 2.1, $|D| = n$. First, we prove that $1 \leq |Y_1 \cap D| \leq 2$ and $1 \leq |Y_n \cap D| \leq 2$. By the symmetry of $G_{n,4}$, we only need to prove that $1 \leq |Y_1 \cap D| \leq 2$. By contradiction. Suppose $|Y_1 \cap D| = 0$ or $|Y_1 \cap D| \geq 3$.

If $|Y_1 \cap D| = 0$, then $|Y_2 \cap D| = 4$. By Lemma 2.2, $|D \cap V(H_{n-2,4})| \geq \gamma(H_{n-3,4})$. By Lemma 2.1, $|D| \geq 4 + \gamma(H_{n-3,4}) \geq 4 + n - 3 = n + 1$, a contradiction with $|D| = n$. Thus $|Y_1 \cap D| \geq 1$.

Assume now $|Y_1 \cap D| \geq 3$. By Lemma 2.2, $|D \cap V(H_{n-1,4})| \geq \gamma(H_{n-2,4})$. By Lemma 2.1, $|D| \geq 3 + \gamma(H_{n-2,4}) \geq 3 + n - 2 = n + 1$, a contradiction with $|D| = n$. Thus $|Y_1 \cap D| \leq 2$. ■

Lemma 5.2 *Let D be a minimum dominating set of $G_{n,4}$. Then $|Y_1 \cap D| = 1$, $|Y_n \cap D| = 1$ for $n \in \{4, 7, 8, 10, 11\}$.*

Proof. By the symmetry of $G_{n,4}$ and by Lemma 5.1, we only need to prove $|Y_1 \cap D| \neq 2$. Suppose, to the contrary, that there exists a minimum dominating set D of $G_{n,4}$ such that $|Y_1 \cap D| = 2$.

If $n \neq 10$ then, by Lemma 2.2, $|D \cap V(H_{n-1,4})| \geq \gamma(H_{n-2,4})$. By Lemma 2.1, $|D| \geq 2 + \gamma(H_{n-2,4}) \geq 2 + n - 1 = n + 1$, a contradiction with $|D| = n$.

Now assume $n = 10$. Let $D' = D \setminus Y_1$. If $Y_2 \cap D \neq \emptyset$, then there exists a vertex $u_{2,j}$ such that $D' \cup \{u_{2,j}\}$ is a dominating set of $H_{n-1,4}$. By Lemma 2.1, $|D| = 2 + |D'| \geq 2 + \gamma(G_{9,4}) - 1 = 11$, a contradiction with $|D| = 10$. Next, we assume that $Y_2 \cap D = \emptyset$ and then $|Y_3 \cap D| \geq 2$. By Lemma 2.2, $|D \cap V(H_{n-3,4})| \geq \gamma(H_{n-4,4})$. By Lemma 2.1, $|D| \geq 4 + \gamma(H_{n-4,4}) = 4 + 7 = 11$, a contradiction with $|D| = 10$. The Lemma follows. ■

Lemma 5.3 *Let D be a minimum dominating set of $G_{n,4}$. Then $|Y_1 \cap D| = 1$, $|Y_n \cap D| = 1$ for $n \notin A$.*

Proof. By the symmetry of $G_{n,4}$ and by Lemma 5.1, we only need to prove $|Y_1 \cap D| \neq 2$. By Lemma 5.2, the statement is true for $n \in \{4, 7, 8, 10, 11\}$. We proceed by induction on $n \geq 12$.

Suppose that the assertion is true for any integer k with $10 \leq k < n$. Suppose, to the contrary, that there exists a minimum dominating set D of $G_{n,4}$ such that $|Y_1 \cap D| = 2$. If $Y_2 \cap D = \emptyset$, then $D' = D \setminus Y_1$ is a dominating set of $H_{n-2,4}$ and $|Y_3 \cap D'| \geq 2$. By the induction hypothesis, D' is not a minimum dominating set of $H_{n-2,4}$, and hence $|D'| \geq \gamma(H_{n-2,4}) + 1 \geq n - 1$ by Lemma 2.1. Then $|D| = 2 + |D'| \geq n + 1$, a contradiction with $|D| = n$.

If $Y_2 \cap D \neq \emptyset$, there exists a vertex $u_{2,j}$ such that $D'' = (D \setminus Y_1) \cup \{u_{2,j}\}$ is a dominating set of $H_{n-1,4}$ and $|Y_2 \cap D''| \geq 2$. By the induction hypothesis, D'' is not a minimum dominating set of $H_{n-1,4}$, and hence $|D''| \geq \gamma(H_{n-1,4}) + 1 \geq n$. Then $|D| \geq 2 + |D''| - 1 \geq n + 1$, a contradiction with $|D| = n$. The Lemma follows. ■

Theorem 5.1 $b(G_{5,4}) = b(G_{9,4}) = 3$, $b(G_{6,4}) = 2$, and $b(G_{n,4}) = 1$ for $n \notin A$.

Proof. It is easy to verify that $b(G_{5,4}) = b(G_{9,4}) = 3$ and $b(G_{6,4}) = 2$. Next, we prove $b(G_{n,4}) = 1$ for $n \notin A$. Then $n \geq 4$. Let D be a minimum dominating set of $G_{n,4} - u_{1,2}u_{1,3}$. By Lemma 2.1, we only need to show that $|D| \geq n + 1$. We prove the conclusion by considering three cases, respectively.

Case 1 $|Y_1 \cap D| = 0$.

Then $|Y_2 \cap D| = 4$. By Lemma 2.2, $|D \cap V(H_{n-2,4})| \geq \gamma(H_{n-3,4})$. Thus $|D| \geq 4 + \gamma(H_{n-3,4}) \geq n + 1$.

Case 2 $|Y_1 \cap D| \geq 2$.

Then D is a dominating set of $G_{n,4}$ with $|Y_1 \cap D| \geq 2$. By Lemma 5.3, D is not a minimum dominating set of $G_{n,4}$, and hence $|D| \geq n + 1$ by Lemma 2.1.

Case 3 $|Y_1 \cap D| = 1$.

Without loss of generality, let $u_{1,j_0} \in D$ and $j_0 \leq 1$. Then $u_{2,3}, u_{2,4} \in D$ and hence $|Y_2 \cap D| \geq 2$.

Let $D' = D \setminus \{u_{1,j_0}\}$. If $j_0 = 2$, or $|Y_2 \cap D| \geq 3$, or $j_0 = 1$ and $u_{31} \in D'$, then D' is a dominating set of $H_{n-1,4}$ and let $D'' = D'$. Assume now $j_0 = 1$, $u_{31} \notin D$, and $Y_2 \cap D = \{u_{2,3}, u_{2,4}\}$. If $u_{3,2}$ or $u_{3,3}$ or $u_{3,4}$ belongs to D , then $D'' = (D' \setminus \{u_{2,3}\}) \cup \{u_{2,2}\}$ is a dominating set of $H_{n-1,4}$ with $|Y_2 \cap D''| \geq 2$.

If $n \in \{4, 7, 10\}$, then $|D''| \geq \gamma(H_{n-1,4}) = n$ by Lemma 2.1. If $n \notin \{4, 7, 10\}$, then D'' is not a minimum dominating set of $H_{n-1,4}$ by Lemma 5.3. By Lemma 2.1, $|D''| \geq \gamma(H_{n-1,4}) + 1 = n$. Thus $|D| \geq |D''| + 1 \geq n + 1$.

In the following assume $j_0 = 1$, $u_{31} \notin D$, $Y_2 \cap D = \{u_{2,3}, u_{2,4}\}$, and $u_{3,2}, u_{3,3}, u_{3,4} \notin D$. Then $u_{4,1}, u_{4,2}$ should be in D to dominate u_{31} and u_{32} and $D''' = D \setminus \{u_{1,1}, u_{2,3}, u_{2,4}\}$ is a dominating set of $H_{n-3,4}$ with $|Y_3 \cap D'''| \geq 2$.

If $n \in \{4, 8, 12\}$, then $|D'''| \geq \gamma(H_{n-3,4}) = n - 2$ by Lemma 2.1. If $n \notin \{4, 8, 12\}$, then D''' is not a minimum dominating set of $H_{n-3,4}$ by Lemma 5.3. Therefore $|D'''| \geq \gamma(H_{n-3,4}) + 1 = n - 2$ by Lemma 2.1. Thus $|D| \geq 3 + |D'''| \geq n + 1$.

The theorem follows. ■

6 Remarks

Through determining the bondage number of $G_{n,m}$ for $2 \leq m \leq 4$, we find that if we delete the vertex $u_{1,1}$, the domination number is invariable. If m increases, the effect of $u_{1,1}$ for the domination number will be smaller and smaller in view of probability. Therefore we expect that $\gamma(G_{n,m} - u_{1,1}) = \gamma(G_{n,m})$ for $m \geq 5$ and we give the following conjecture.

Conjecture 6.1 $b(G_{n,m}) \leq 2$ for $m \geq 5$.

In our method, determining the bondage number of a graph strongly depends on the domination number of the graph. Even the domination number of some graph, determining its bondage number is also very difficult. For example, the domination number of $G_{n,m}$ for $m = 5$ or 6 has been determined [3, 4], we can not determined its bondage number in our method since the cases that we need to consider are much too. Thus, if we want to determine the bondage number of $G_{n,m}$ for $m \geq 5$ or to solve the Conjecture 6.1, we need a new method except for determining the domination number of $G_{n,m}$ for $m \geq 7$. It is what we further study work.

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